Computational Physics and Methods

A characteristic discontinuous Galerkin scheme for tracer transport

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A new *characteristic discontinuous Galerkin* (CDG) advection scheme is presented. In contrast to standard discontinuous Galerkin schemes, the test functions themselves follow characteristics in order to ensure conservation and the edges of each element are also traced backwards along characteristics in order to create a *swept region*, which is integrated in order to determine the mass flux across the edge. Both the accuracy and performance of the scheme are greatly improved by the use of large CFL numbers for a shear flow test case and the scheme is shown to scale sublinearly with the number of tracers being advected, outperforming a standard flux corrected transport scheme for 10 or more tracers with a linear basis. Moreover the CDG scheme may be run to arbitrarily high order spatial accuracy and on unstructured grids, and is shown to give the correct order of error convergence for piecewise linear and quadratic bases on regular quadrilateral and hexahedral planar grids. Using a modal Taylor series basis, the scheme may be made monotone while preserving conservation with the use of a standard slope limiter, although this reduces the formal accuracy of the scheme to first order. The second order scheme is roughly as accurate as the incremental remap scheme with nonlocal gradient reconstruction at half the horizontal resolution. The scheme is being developed for implementation within the MPAS Ocean model, with the intention that it ultimately be deployed within other MPAS cores as well.

Background and Motivation

Tracer advection constitutes a large portion of the compute time for modern global climate models, due to the large number of chemical and hydrometeor species that must be accounted for. For physical consistency, the advection of tracers must be conservative while being as numerically accurate and computationally efficient as possible. The preservation of monotonicity may also be required. Here we present a novel characteristic discontinuous Galerkin (CDG) advection scheme which allows for arbitrarily long time steps and monotonic, conservative advection on unstructured grids while also scaling sub-linearly with the advection of additional tracers such that it outperforms a traditional flux corrected transport (FCT) scheme for a moderate number of tracers. The scheme may be implemented at arbitrarily high order, through the use of a modal basis expansion of the tracer in each element. [1]

Description

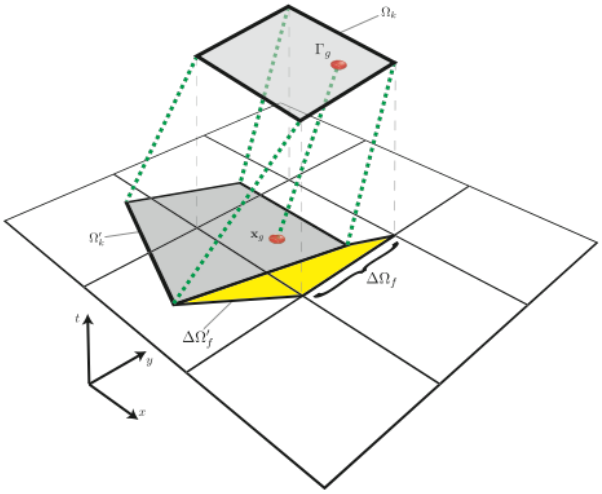
The CDG scheme is formulated by multiplying the standard advection equation for a tracer q by a series of i test functions for each cell k $\varphi_{k,i}(\vec{x},t)$ as:

$$\frac{\partial \varphi_{k,i} q}{\partial t} + \nabla.(\varphi_{k,i}\,\vec{u}q) = \varphi_{k,i}\left(\frac{\partial q}{\partial t} + \nabla.(\vec{u}q)\right) + q\left(\frac{\partial \varphi_{k,i}}{\partial t} + \vec{u}.\nabla\varphi_{k,i}\right).$$

The first term on the right hand side is just the original advection equation and so is zero, while the second term on the right hand side can also be made to go to zero if the test functions themselves are conserved along characteristics. Integrating in space and time and assuming that the tracer is represented by a series of trial functions $q(\vec{x}) \approx \beta_i(\vec{x})c_{k,i}$, where $\beta_{k,i}(\vec{x}) = \varphi_{k,i}(\vec{x}, t^{n+1})$ such that the test functions *arrive* at the trial function locations at the new time level, leads to the Galerkin approximation

$$\int_{\Omega_k} \beta_{k,i}\beta_{k,j}\, d\Omega_k c_{k,j}^{n+1} = \int_{\Omega_k} \varphi_{k,i} q^n d\Omega_k - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_k} \varphi_{k,i}\, \vec{u} q^n . d\vec{s} dt$$

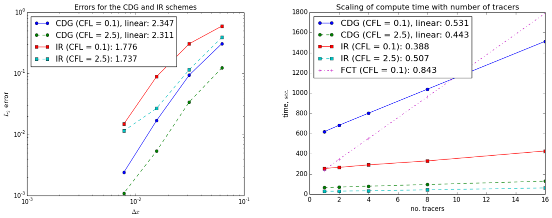
The above expression represents the CDG advection equation. The second term on the right hand side is evaluated by tracing the edges backwards along velocity characterisitcs and integrating the swept region formed by the edge and its pre-image at the previous time level, with the quadrature points themselves integrated forwards in time along those same characteristics.



Flux computation for the CDG scheme. Edge vertices are integrated back in time to their departure points in order to determine the swept region for an edge over a given time step. The volume of the tracer over the swept region is then integrated with the quadrature points integrated forwards to arrival points where the test functions are evaluated in order to preserve the value of the test functions along characteristics.

Anticipated Impact

Because the swept regions of the edges are common for all tracers, the CDG scheme scales sub-linearly with the number of tracers, and outperforms a standard FCT scheme for 10 tracers and beyond on a planar mesh. Moreover because the swept regions are evaluated in a Lagrangian form there is no time step stability constraint and the scheme is more efficient than the FCT scheme for a single tracer with a CFL number of 2.5. The scheme may also be run to high order by extending the orders of the trial and test functions and the quadrature points.



Left: convergence rates for the linear basis CDG and incremental remap (IR) schemes with CFL numbers of 0.1 and 2.5. Both schemes see improved accuracy with increased CFL number due to the reduces number of integrations, while the CDG scheme is approximately as accurate as the IR scheme with half the horizontal resolution due to the compact representation of the trial functions within each cell. Right: performance scaling of the CDG and IR schemes with the number of tracers. With the same CFL number the CDG scheme outperforms the FCT scheme for 10 tracers, while with a CFL of 2.5 it outperforms the FCT scheme for a single tracer.

Path Forward

The CDG scheme is currently being implemented within the MPAS-Ocean model using an operator splitting approach to decouple the horizontal and vertical advection. Once it has been successfully deployed its use in other MPAS cores will also be investigated.

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References

[1] D. Lee, R. Lowrie, M. Petersen, T. Ringler and M. Hecht, "A High Order Characteristic Discontinuous Galerkin Scheme for Advection on Unstructured Meshes," *Journal of Computational Physics*, 2016. doi: 10.1016/j.jcp.2016.08.010