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Title: Tropical Matrix Factorization

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Tropical Matrix Factorization

Jose Ortiz

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Introduction

- Replacing addition with maximum (or minimum) and swapping multiplication with addition gives the **tropical numbers** $\mathbb{T} = (\overline{\mathbb{R}}, \oplus, \odot)$.

Classical Setting ⟷ ***Tropical Setting***

Classical Setting	Tropical Setting
$1 + 0 = 1$	$1 \oplus 0 := \max(1, 0) = 1$
$1 + 1 = 2$	$1 \oplus 1 := \max(1, 1) = 1$
$1 + 2 = 3$	$1 \oplus 2 := \max(1, 2) = 2$
$1 \cdot 0 = 0$	$1 \odot 0 := 1 + 0 = 1$
$1 \cdot 1 = 1$	$1 \odot 1 := 1 + 1 = 2$
$3 \cdot 3 = 9$	$3 \odot 3 := 3 + 3 = 6$

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- Compared to addition and multiplication, comparison and addition are **faster**.
- Comparison and addition are **robust** operations.
- Moving between classical and tropical settings is often **reversible**.
  - Classical $\rightarrow$ Tropical: *Easy*
  - Classical $\leftarrow$ Tropical: *Harder*

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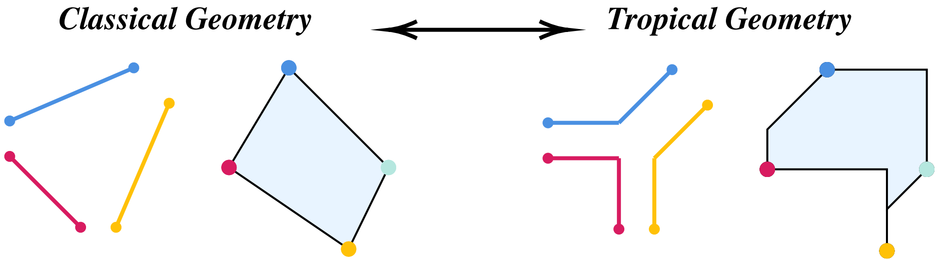

$ax + by + c$ $\max(a + x,\ b + y,\ c)$

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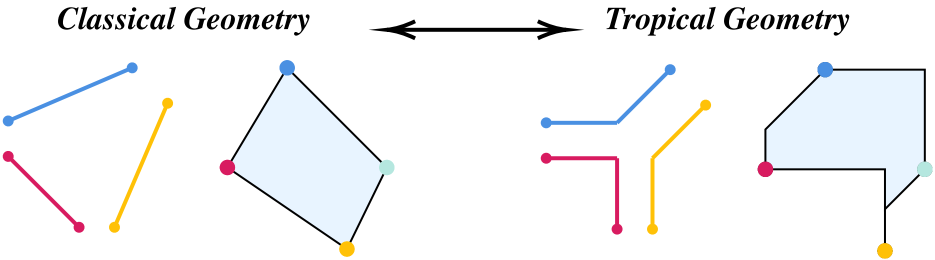

$ax + by + c$ $\max(a + x,\ b + y,\ c)$

- This project was an exploration of how tropical methods could be applied to inexact computing.

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Tropical Matrices

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$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \end{bmatrix} \odot \begin{bmatrix} \mathbf{5} & 3 \\ \mathbf{2} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{max}(\mathbf{1}+\mathbf{5}, \mathbf{2}+\mathbf{2}) & \max(1+3, 2+0) \\ \max(3+5, 4+2) & \max(3+3, 4+0) \end{bmatrix} = \begin{bmatrix} \mathbf{6} & 4 \\ 8 & 6 \end{bmatrix}$$

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- *Many* entries could be changed without affecting the product.
- The usual factorization techniques **don't work here**.

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Matrix Factorization

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    - Modification of `TMF` that introduces sparsity
    - Applied to matrix completion and prediction tasks
    - Convergence and complexity are inherited from `TMF`
- Both are *very* slow, but great at capturing non-linear structures.

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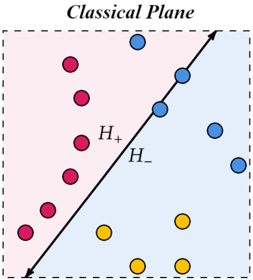
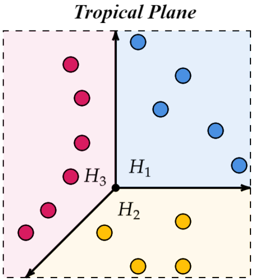
- Tropical methods can **encode data with 'extreme' values**.
- Tropical hyperplanes separate the ambient space into $n+1$ regions–**encoding more information with fewer parameters**.





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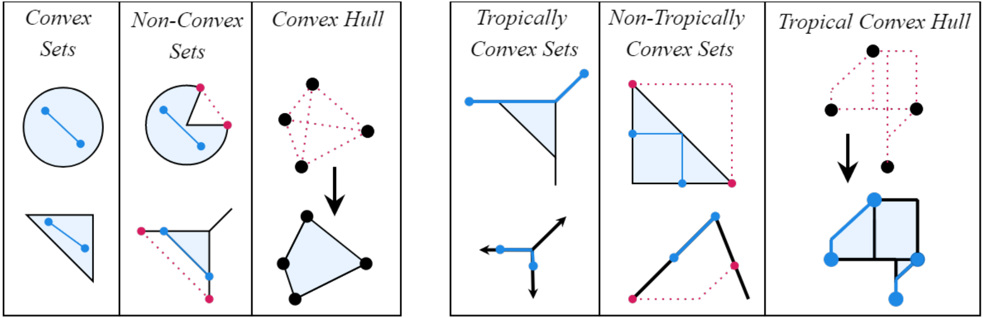
A Geometric Approach to Matrix Factorization

- Key observation:
  - *The columns (or rows) of M can be identified with points that generate a* **tropical convex hull** *tconv*(M).



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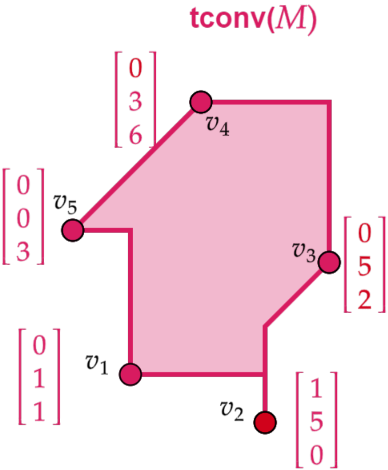
- The left factor A that results from this method is **stable under translations** of M; only B varies as M is perturbed.
- This approach **can be modified** such as with `STMF` in order to be applied to prediction tasks.
- This approach has a clear **geometric interpretation**.

Overview of the Geometric Method

- The columns of M can be identified with the tropical convex hull tconv(M).

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 5 & 5 & 3 & 0 \\ 1 & 0 & 2 & 6 & 3 \end{bmatrix} \quad \leftrightsquigarrow$$



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- Which is contained in some tconv(A).

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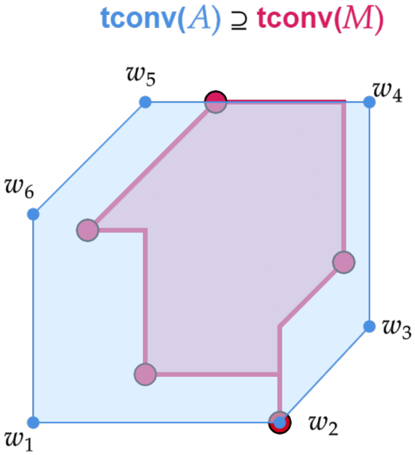

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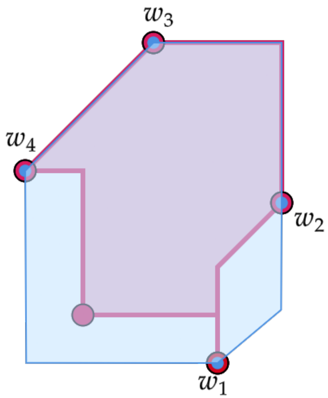
tconv(A) ⊇ tconv(M)



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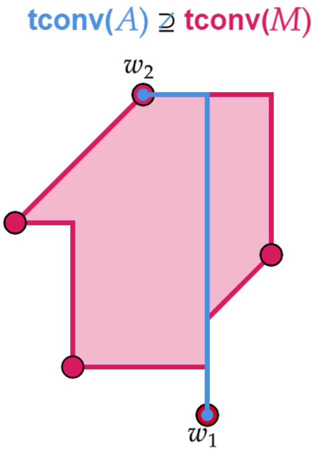

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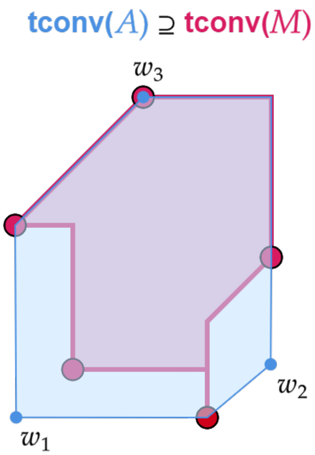

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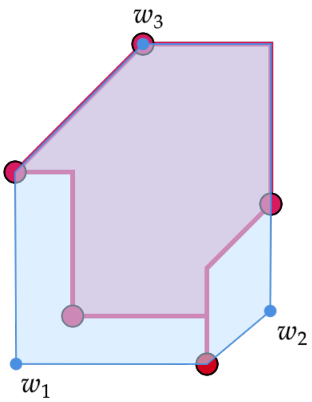

[2]

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$$\underset{w_1\, w_2\, w_3}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}} \odot \begin{bmatrix} -1 & 0 & -1 & -1 & -1 \\ 4 & 0 & 0 & -2 & -5 \\ -5 & -6 & -4 & 0 & -3 \end{bmatrix} = \underset{v_1\, v_2\, v_3\, v_4\, v_5}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 5 & 5 & 3 & 0 \\ 1 & 0 & 2 & 6 & 3 \end{bmatrix}}$$

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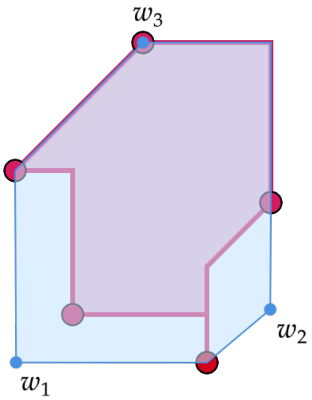

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Performance Results

- Running TMF and GTMF 100 times on random 6×6 matrices:

	TMF	GTMF
Time	3.68884	0.00168
Error	5.18909	$1.21192 \cdot 10^{-14}$
Added Error	0.69285	0.00039
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- Questions?

Thank You!

Thanks to my mentor Laura Monroe for her guidance and to team members Vanessa Job, Nathan Kodama, Andrew Alexander, and Matthew Broussard for their helpful suggestions and questions that led me to this problem.

Email: joseaortiz@ksu.edu

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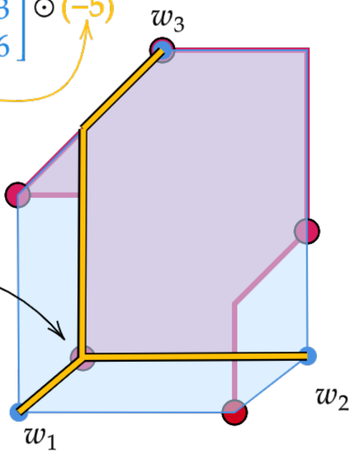
$$\underset{w_1\,w_2\,w_3}{\begin{bmatrix}1&0&0\\1&5&3\\0&0&6\end{bmatrix}} \odot \begin{bmatrix}-1\\-4\\-5\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} \odot (-1) \oplus \begin{bmatrix}0\\5\\0\end{bmatrix} \odot (-4) \oplus \begin{bmatrix}0\\3\\6\end{bmatrix} \odot (-5)$$

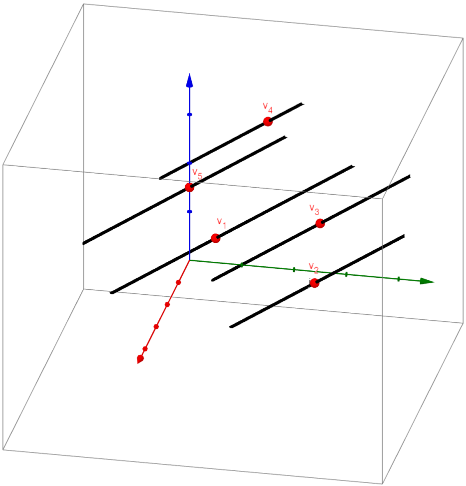
$$= \underset{v_1}{\begin{bmatrix}0\\1\\1\end{bmatrix}}$$

'Lattice distance'

$$A \odot B = M$$

$$\underset{w_1\,w_2\,w_3}{\begin{bmatrix}1&0&0\\1&5&3\\0&0&6\end{bmatrix}} \odot \begin{bmatrix}-1&0&-1&-1&-1\\-4&0&0&-2&-5\\-5&-6&-4&0&-3\end{bmatrix} = \underset{v_1\;v_2\;v_3\;v_4\;v_5}{\begin{bmatrix}0&1&0&0&0\\1&5&5&3&0\\1&0&2&6&3\end{bmatrix}}$$





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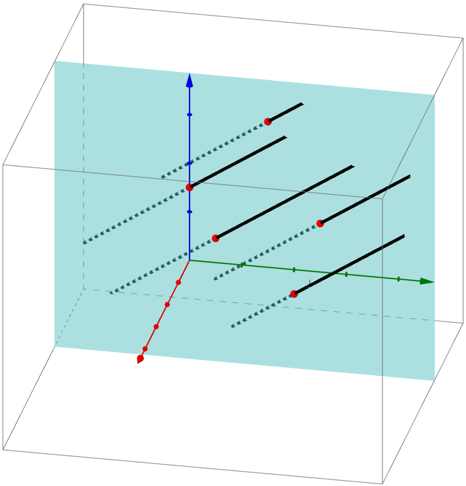

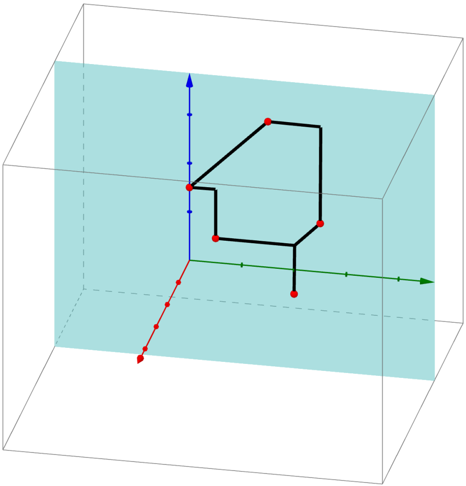

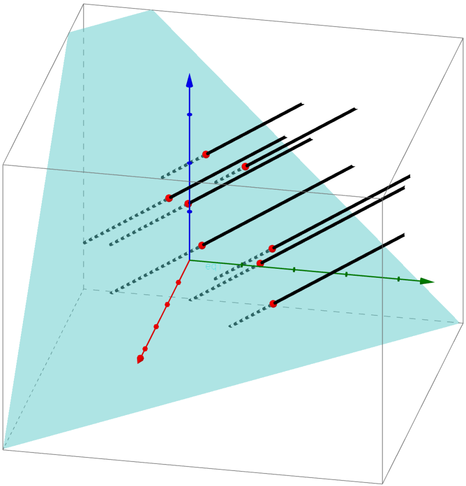

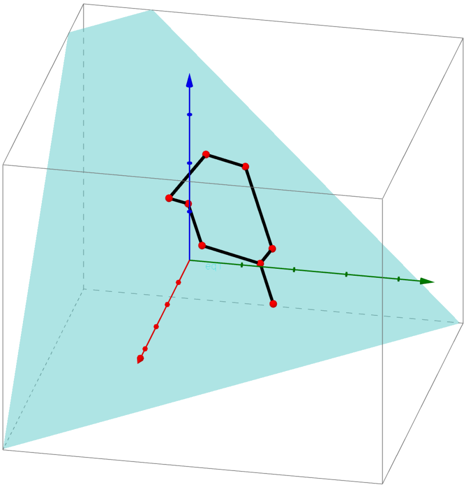

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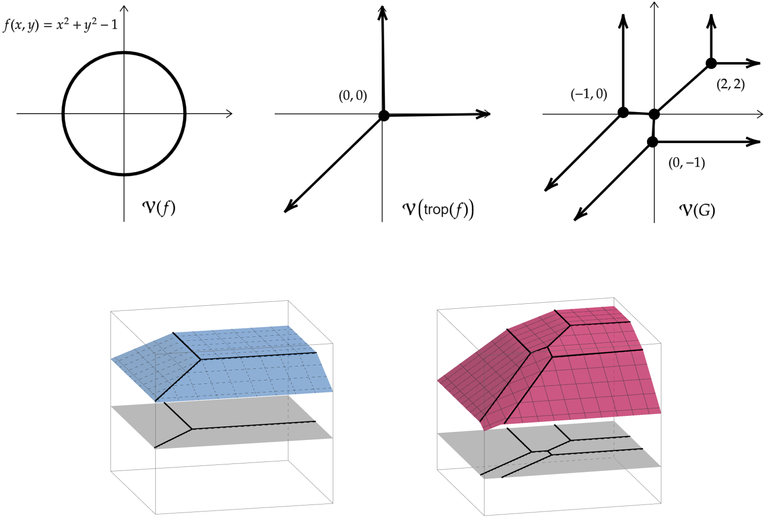
$f(x, y) = x^2 + y^2 - 1$
$\mathcal{V}(f)$
(0, 0)
$\mathcal{V}(\mathrm{trop}(f))$
(−1, 0)
(2, 2)
(0, −1)
$\mathcal{V}(G)$